Physics 618 2020

- U(1) bundles / symplectic tori
- Rep. of Heisenberg given no canonical Lag. subgroup
- Induced representations
- Lifting symplectic automorphisms
- $Heis(\mathbb{R}^2)$ and $SL(2,\mathbb{R})$

May 1, 2020

Heisenberg construction

"$U(1)$ principal bundles / symplectic tori"

$\Gamma \subset \mathbb{R}^n$ embedded lattice of full rank

$\mathbb{R}^n / \Gamma = T$ torus.

$\mathbb{C} / (\mathbb{Z} + \tau\mathbb{Z}) = E_\tau$ elliptic curve.

Suppose also that we're given a bilinear form on Γ

$$\Omega : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$$

Γ = lattice $\cong \mathbb{Z}^n$

Ω antisymmetric.

Seek to put them in can. form.

$S \in \mathrm{Aut}(\Gamma) \cong GL(n, \mathbb{Z})$

if we choose ordered basis $\gamma_1 \cdots \gamma_n$ for Γ

$$\Omega(\gamma_i, \gamma_j) = \Omega_{ij}$$ anti-symm matrix of integers

$$\Omega \longrightarrow S^{tr} \Omega S.$$

Thm: $\exists\, S$

$$\Omega_{ij} = \left(\begin{array}{cc|cc|c|c} 0 & d_1 & & & & \\ -d_1 & 0 & & & & \\ \hline & & 0 & d_2 & & \\ & & -d_2 & & & \\ \hline & & & & \ddots & \\ \hline & & & & & 0 \cdots 0 \end{array}\right)$$

$d_i \in \mathbb{Z}$

We'll assume Ω_{ij} nondegenerate

$\Longrightarrow$ n is even

$$\Omega_{ij} = \begin{pmatrix} \begin{array}{cc|} 0 & d_1 \\ -d_1 & 0 \\ \hline \end{array} & & \\ & \ddots & \\ & & \begin{array}{|cc} \hline 0 & d_s \\ -d_s & 0 \end{array} \end{pmatrix}$$

$d_i \neq 0$

extending from Γ to $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} = V$

we get a nondegenerate bilinear form on V. : Symplectic form

$$\Omega(x\gamma + y\gamma', \gamma'') \qquad \gamma, \gamma' \in \Gamma$$

$$:= x\,\Omega(\gamma, \gamma'') + y\,\Omega(\gamma', \gamma'')$$

$x, y \in \mathbb{R}$ $\qquad \Gamma \rightsquigarrow V$.

Define commutator function on V

$$k(v_1, v_2) = e^{2\pi i\,\Omega(v_1, v_2)}$$

$$1 \to U(1) \to \mathrm{Heis}(V,\Omega) \xrightarrow{\pi} V \to 0$$

Choose cocycle $f(v_1, v_2) = e^{i\pi\Omega(v_1,v_2)}$

$$(z_1, v_1)\cdot(z_2, v_2) = \left(z_1 z_2 e^{i\pi\Omega(v_1,v_2)}, v_1+v_2\right)$$

does not split.

Pullback to Γ <u>does</u> split.

$$\begin{array}{c} \mathrm{Heis}(V,\Omega) \\ \cup \end{array}$$

$$1 \to U(1) \to \pi^{-1}(\Gamma) \underset{\curvearrowleft s}{\xrightarrow{\pi}} \Gamma \to 0$$

$s(\gamma) = (\epsilon_\gamma, \gamma)$ will split if

$$\epsilon_{\gamma_1}\epsilon_{\gamma_2} = e^{-i\pi\Omega(\gamma_1,\gamma_2)}\epsilon_{\gamma_1+\gamma_2}$$

$$U(1) \to \mathrm{Heis}(V,\Omega)/s(\Gamma) \xrightarrow{\pi} \mathbb{T}$$

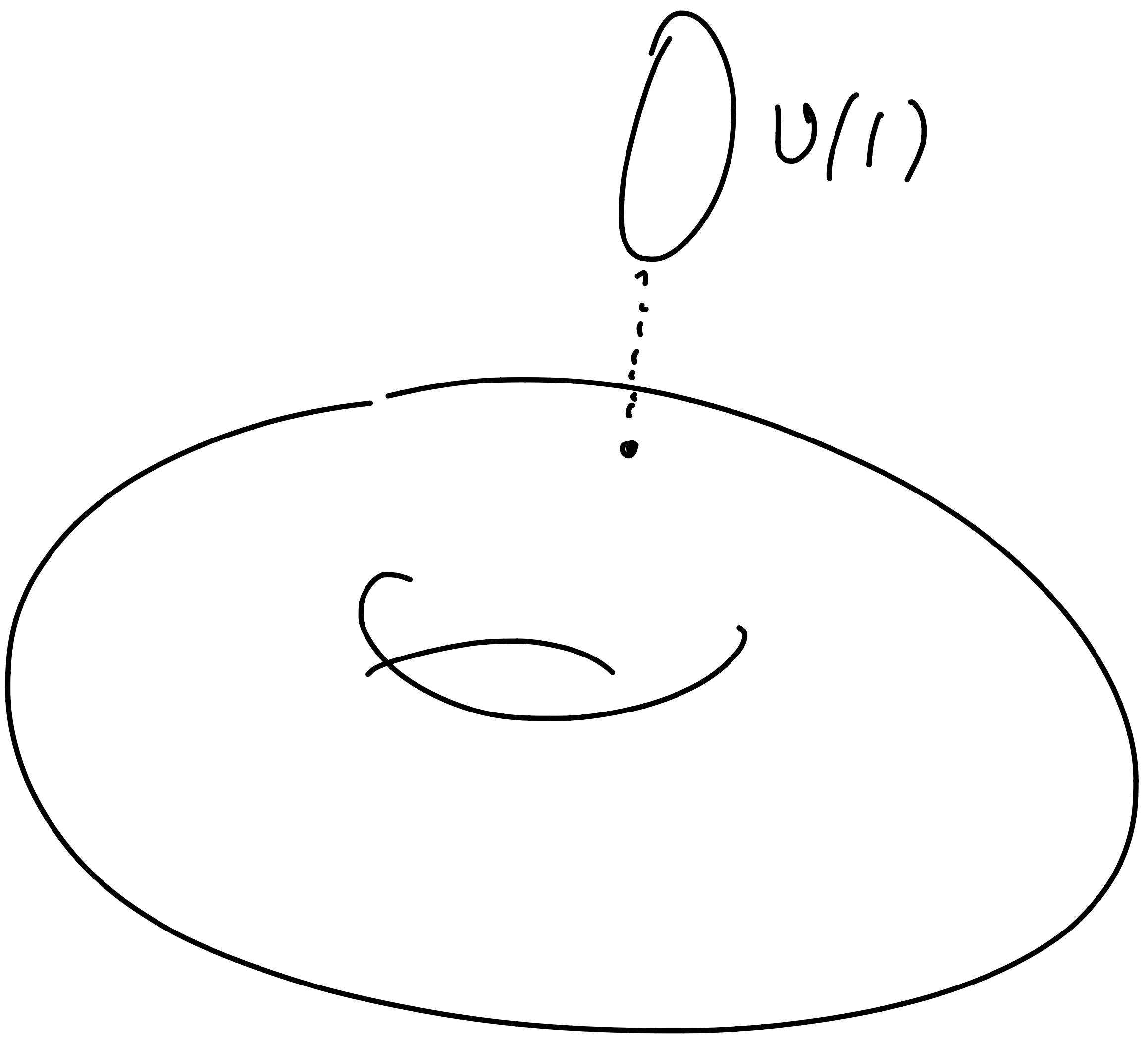
$$U(1) \cong \pi^{-1}(v \bmod \Gamma)$$

$$P = (U(1)\times V)/\Gamma$$

$$(z, v) \sim (z, v)\cdot(\epsilon_\gamma, \gamma)$$

$$= \left(z\epsilon_\gamma e^{i\pi\Omega(v,\gamma)}, v+\gamma\right)$$

$$[(z,v)] \xrightarrow{\pi} [v] = [v+\gamma]$$



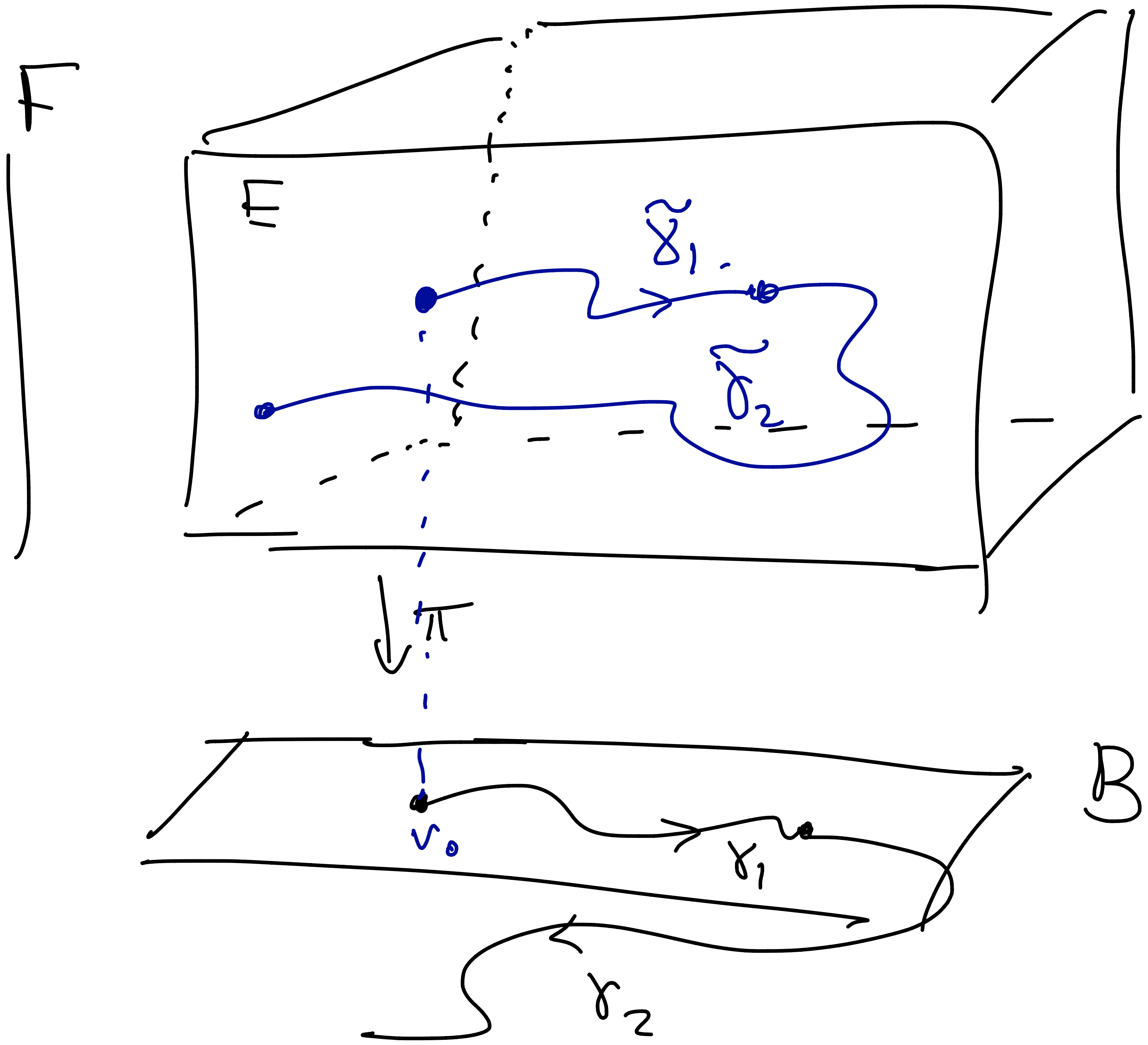
$= E_\tau$

↳ Connection

$$1 \to U(1) \to \mathrm{Heis}(V,\Omega) \xrightarrow{\pi} V \to 1$$

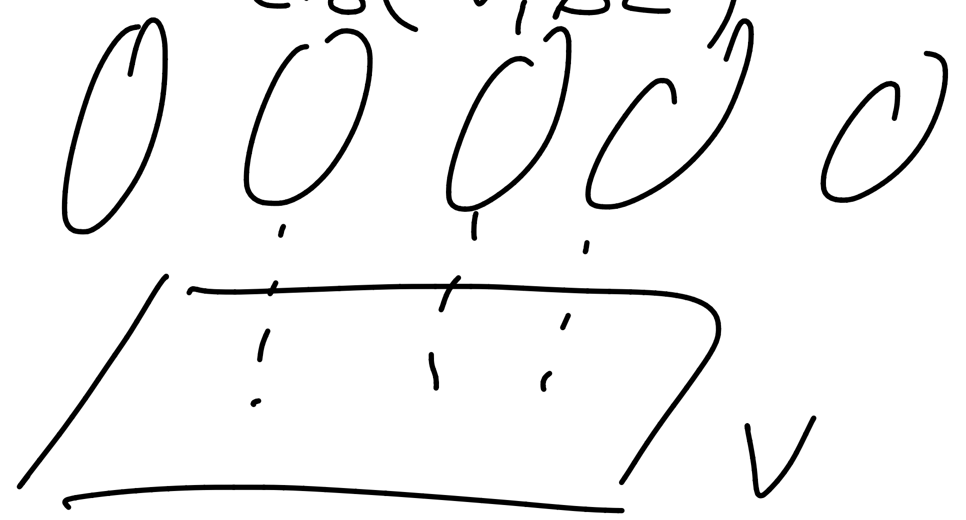
principal $U(1)$ bundle.

Connection



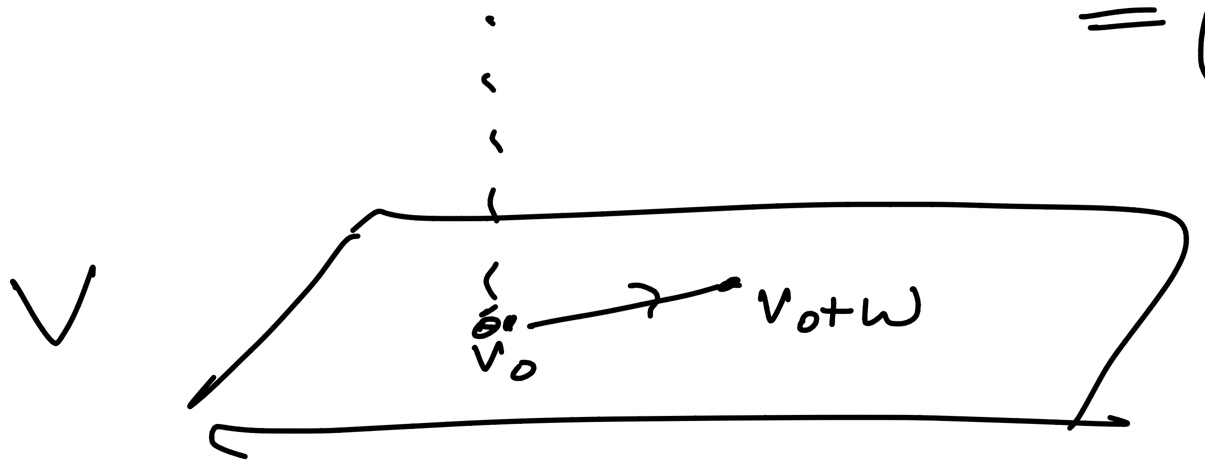
In our example it suffices to define the connection on straightline paths.

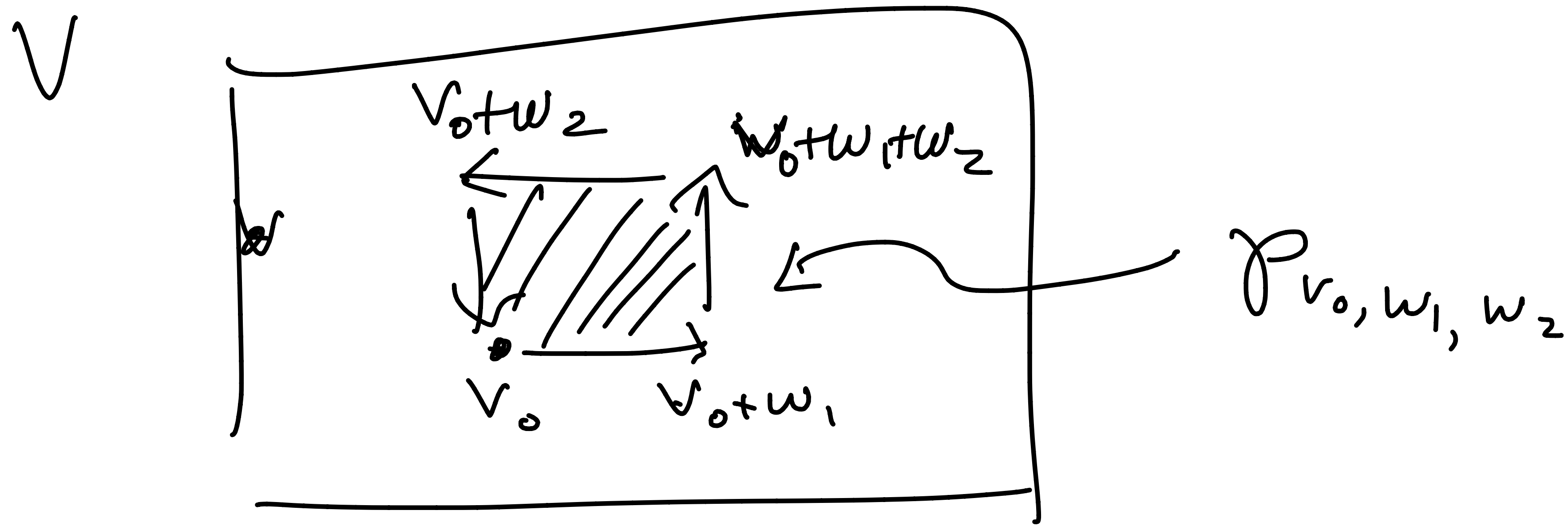
$$\mathrm{Heis}(V, \Omega) \longrightarrow V$$



$\gamma_{v_0, w} = \{ v_0 + tw \mid 0 \leq t \leq 1 \}$

$(z, v_0) \xrightarrow{U} (1, w) \cdot (z, v_0)$
$= \left(z e^{i\pi \Omega(w, v_0)}, w + v_0\right)$





$U : (z, v_0) \longrightarrow \left(z e^{2\pi i \Omega(w_1, w_2)}, v_0\right)$

$\Longrightarrow$ Connection has curvature

Descends to $U(1)$ bundle over T

$[\Omega]$ nontrivial coho. class.

$d_1, \dots, d_s$ these give first Chern class of the bundle.

Rep. of Heis. when we don't have it presented as Heis$(S \times \hat{S})$

$$
\begin{array}{ccccccccc}
1 \to & Z & \to & \tilde{G} & \to & G & \to 1 \\
& \cap & & & & \uparrow & \\
& U(1) & & & & \text{Abelian} &
\end{array}
$$

k – comm. function : analog of symplectic form

$L \subset G$ Lagrangian if $k|_{L \times L} = 1$

$k(g_1, g_2) = 1 \quad \forall\, g_1, g_2 \in L.$

For $G = \underline{\underline{S \times \hat{S}}}$

$(S \times 1) \times (1 \times \hat{S})$ are max. Lag. sub

In general G not presented as $L \times L'$

How to represent Heis ?

For Heis $(S \times \hat{S})$

SvN : $L^2(S)$ or $L^2(\hat{S})$.

<u>Choose</u> some max. Lag. $L \subset G$

$\pi^{-1}(L) = \tilde{L} \subset \mathrm{Heis}(G)$ maxi Abeli. Subgroup.

<u>Choose</u> character of $\tilde{L}$

$\nu : \tilde{L} \to \mathbb{C}$ $\quad \nu(z,x) = z\,\underbrace{\nu(x)}_{\text{char. on } L}$.

$$\left[\begin{array}{l} \nu(x)\nu(x') = f(x,x')\,\nu(x+x') \\ \forall\, x, x' \in L \end{array}\right.$$

V = Rep. Space = Space of all functions $\mathrm{Heis}(G) \longrightarrow \mathbb{C}$ that are <u>equivariant</u> under Right action by $\tilde{L}$

$\Psi \in V$ means

$\underline{\Psi} : \tilde{G} = \mathrm{Heis}(G) \longrightarrow \mathbb{C}$

s.t. $\forall\, (z',x') \in \tilde{L} \subset \tilde{G}$

$$\boxed{\underline{\Psi}\Big((z,x)\cdot(z',x')\Big) = \frac{1}{\nu(\cancel{z}',x')}\,\underline{\Psi}(z,x)}$$

$$\underline{\Psi}(z,x) = \frac{1}{z}\,\underline{\Psi}(1,x)$$

$$\Psi(x) := \underline{\Psi}(1,x)$$

$$\Psi : G \longrightarrow \mathbb{C}$$

$$\boxed{\begin{gathered} \Psi(x+x') = \frac{f(x,x')}{\nu(x')}\,\psi(x) \\ \forall x' \in L,\ x \in G \end{gathered}}$$

$|\psi(x)|^2$ descends to G/L

$$\int_{G/L} |\psi(x)|^2 < \infty.$$

Notes If $G = \mathrm{Heis}(S \times \hat{S})$

$L = 1 \times \hat{S}$

Recover $V \cong L^2(S)$ w/ SvN rep

$$1 \to \mathbb{F}_2 \to \underset{\mathcal{E}_{2m}}{\overset{\shortparallel}{\mathrm{Heis}(\mathbb{F}_2)}} \longrightarrow \mathbb{F}_2^{2m} \longrightarrow 1$$

$$\mathbb{F}_2 = \mathbb{Z}_2 \qquad k(w, w') = (-1)^{\sum_{i \neq j} w_i w_j'}$$

$$L \subset \mathbb{F}_2^{2m} = L \oplus N$$

$\nwarrow$ Lagrangian $\qquad l \in T_l \quad n \in N \quad x_n$

$l = (0, 1, 0, 0, \cdots 0)$

Can construct such Lag. subspaces using classical error-correcting codes.

N is a group of characters on L

$$\chi_n(\ell) = k(n, \ell)$$

$$M_\chi \rightsquigarrow M_n$$

$L \cong \mathbb{F}_2^m$ $\quad$ $\mathrm{Fun}(L \to \mathbb{C})$

is a 2^m dim'l cplx v.s.

T_ℓ, M_n in terms of γ_i.

Induced Representations

G – group, we want to give rep's of G.

$H \subset G$ subgroup.

Easy to give rep's of H.

Choose a rep. $\rho : H \longrightarrow GL(V)$

vector space.

Then we get a canonical rep$\underline{n}$ of G called the induced rep denoted $\mathrm{Ind}_H^G(V)$

$$\mathrm{Ind}_H^G(V) = \left\{ \begin{array}{c} H\text{- equivariant functions} \\ \Psi : G \longrightarrow V \end{array} \right\}$$

$$\Psi \in \mathrm{Map}(G \longrightarrow \underset{\underset{H}{\circlearrowleft}}{V})$$

$G \times H$ action on this fn. space

$$\big((g,h)\cdot\Psi\big)(g_0) := \underline{\underline{\rho(h)}} \cdot \underline{\Psi(g^{-1}g_0 h)}$$

valued in V

you check: well-defined group action.

Look @ subspace of fixed vectors under $1 \times H$:

$$\rho(h)\,\Psi(g_0 h) = \Psi(g_0)$$

i.e.

$$\boxed{\begin{gathered}\Psi(g_0 h) = \rho(h^{-1})\,\Psi(g_0)\\ \forall\, g_0 \in G,\ h \in H\end{gathered}} \quad (*)$$

"H-equivariant functions"

$$(g\cdot\Psi)(g_0) := \Psi(g^{-1}g_0)$$

If Ψ is H-equiv. then $g\cdot\Psi$ is H-equiv!

$\therefore$ $\mathrm{Ind}_H^G(V)$ is a rep of G.

Geometrical Interpretation

$G \times V$ This is a V bundle over G.

Make a More interesting vector bundle:

(right) H-action on $G \times V$

$$\phi_h : (g,v) \longrightarrow (g\cdot h, \rho(h^{-1})\cdot v)$$

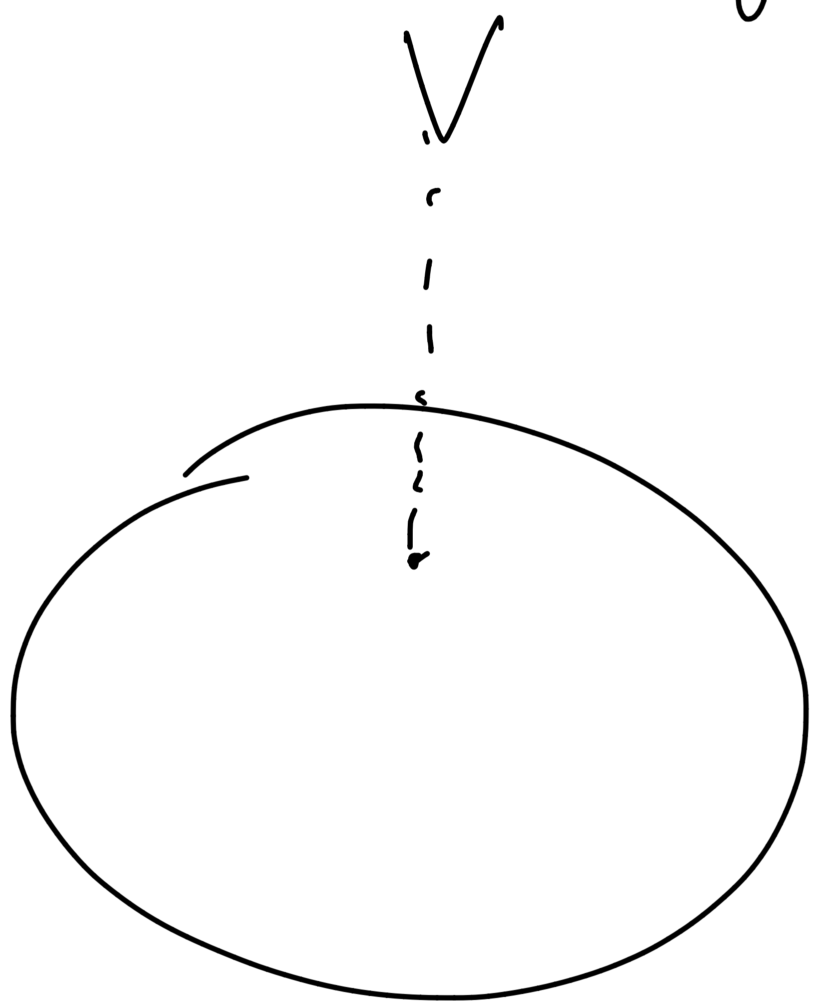
$G \times_H V$ = set of equiv. classes

$\downarrow \pi$ fiber is V

G/H π is cont.

$\pi([g,v]) = gH$

ρ cont.
G, H Lie groups
$\Rightarrow \pi$ cont.



G/H

$\pi^{-1}(gH)$ = copy of V $\Longrightarrow$

Homogeneous vector bundle / G/H

Section $\quad s: G/H \longrightarrow G\times_H V$

$\pi \circ s = Id_{G/H}$.

$$s(gH) = [(g, v(g))]$$

$$= \{(gh, \underline{\underline{v(gh)}}) \mid h \in H\}$$

$$(gh, v(gh)) \sim (g, v(g))$$

but

$$(gh, v(gh)) \overset{\text{def. of eq. rel.}}{\sim} (ghh^{-1}, \rho(h)v(gh))$$

$$= (g, \rho(h)\, v(gh))$$

So

$$\boxed{v(gh) = \rho(h^{-1})\, v(g)}$$

H - equivariance condition!!

$$\left\{ \text{sections of the homogeneous bundle } \pi : G \times_H V \longrightarrow G/H \right\}$$

$\cong$ $\hookrightarrow$ 1-1 correspondence

$$\left\{ \begin{array}{l} \text{equivariant functions} \\ \quad \psi : G \longrightarrow V \\ \psi(gh) = \rho(h^{-1})\, \psi(g) \end{array} \right\}$$

<u>Example:</u>

Reps of $SU(2) \supset \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \right\}$
$\qquad\qquad\qquad\qquad \cong U(1)$

$\widehat{U(1)} = \mathbb{Z}$

$$\rho_k : \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \mapsto e^{ik\theta}$$

$V \cong \mathbb{C}$

Consider the induced repn. of $SU(2)$

$$SU(2)/U(1) \cong S^2 \cong \mathbb{CP}^1$$

$G \times_H V$ is a complex line bundle over S^2 or $\mathbb{CP}^1$.

$\Gamma(\mathcal{L} \to \mathbb{CP}^1) =$ equiv. functions.

$$\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in SU(2)$$

$$\Psi\left(\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \right)$$

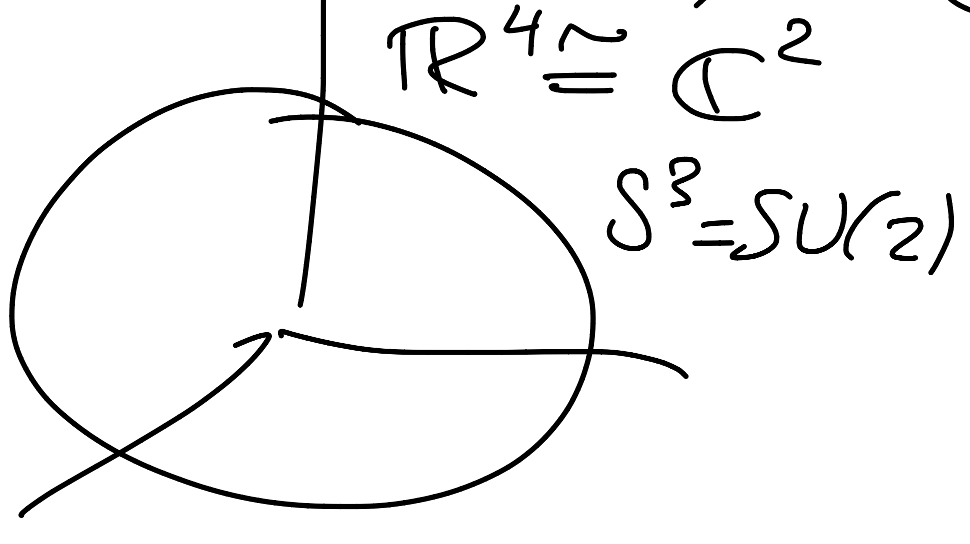
$$= \underbrace{e^{-ik\theta}}_{\rho(h^{-1})} \cdot \Psi \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}$$

$\Psi(u, v)$ = simpler notation.

$$\boxed{\Psi(ue^{i\theta}, ve^{i\theta}) = e^{-ik\theta}\,\Psi(u, v)}$$

∞-diml space of such functions on $SU(2) = S^3$.

But



Look @ the subspace of holomophic functions on $\mathbb{C}^2$ which have been restricted to

$$SU(2) = S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$$

In other words: just drop the condition $|u|^2 + |v|^2 = 1$

$\psi(u,v)$ <u>holo</u>

$$\boxed{\psi(ue^{i\theta}, ve^{i\theta}) = e^{-ik\theta}\psi(u,v)}$$

ψ has to be homogeneous together w/ holomorphy $\Rightarrow$ polynomial and $-k \geq 0$

$$-k = 2j \in \mathbb{Z}_+$$

$$j \in \{0, 1/2, 1, 3/2, \dots\}.$$

Inside the induced repⁿ
the holomorphic equiv. functions
form the vector space

$$\mathcal{H}_{2j} = \left\{ \begin{array}{l} \text{homog. polynom's} \\ \text{in } u, v \text{ of degree } 2j \end{array} \right\}.$$

Choose a basis:

$$\tilde{f}_m(u,v) = u^{j+m} v^{j-m}$$

$$m = -j, -j+1, -j+2, \cdots, j-1, j$$

$(2j+1)$-diml repⁿ.

Let's see how $SU(2)$ is rep. on $\mathcal{H}_{2j}$.

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

Working through def. of the left G-action on $\mathrm{Ind}_H^G(V)$

$$(g \cdot \tilde{f}_m)(u,v) = \tilde{f}_m(\bar{\alpha}u + \bar{\beta}v, -\beta u + \alpha v)$$

$$= (\bar{\alpha}u + \bar{\beta}v)^{j+m}(-\beta u + \alpha v)^{j-m}$$

$$= \sum_{m'} \tilde{D}^j_{m',m}(g) \cdot \tilde{f}_{m'}(u,v)$$

$$\tilde{D}^j_{m',m}(g) = \sum_{s+t=j+m'} \binom{j+m}{s}\binom{j-m}{t} \bar{\alpha}^s \alpha^{j-m-t} \bar{\beta}^{j+m-s} (-\beta)^t$$

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

$\tilde{D}$ - explicit function on $SU(2)$

Matrix elements of irreps form complete set of functions on $L^2(G)$

So $\tilde{D}^j_{m',m}$ complete orthog. basis for $L^2(SU(2))$

Wigner functions. (~~N~~ not quite normalized)

Special cases — spherical harmonics
associated Legendre

$$\tilde{f}_m \underset{\substack{\text{normalization}\\ \text{change}}}{\rightsquigarrow} \underbrace{|m\,j\rangle}_{\text{standard QM.}}$$

$$\frac{1}{\sqrt{(j-m)!\,(j+m)!}}$$

Similarly one can derive the unitary irreps of the Lorentz + Poincaré groups using induced rep$\underline{^{n}}$'s.

$L^2(G)$ is a rep of $G_L \times G_R$

$$\left(\rho(g_L, g_R)\cdot\psi\right)(g_0) = \psi(g_L^{-1} g\, g_R)$$

$D^j_{m',m}$ is a rep of $SU(2)\times SU(2)$

it is the (j,j) rep.

$$D^j_{m',m}(g_L^{-1} g_0\, g_R) \leftarrow$$

$$= \underline{D^j_{m'm''}(g_L^{-1})}\, D^j_{m'',\tilde{m}}(g_0)\, .$$

$$\underline{D^j_{\tilde{m}m}(g_R)}$$

$$L^2(SU(2)) \cong \bigoplus_{j=0}^{\infty} \bar{V}_j \otimes V_j$$

The space of <u>all</u> functions on $SU(2)$ with H-equiv. condition

$$\psi\left(g\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta}\end{pmatrix}\right) = e^{-ik\theta}\,\psi(g)$$

Spanned by

$$D^{l}_{m,-k}(g) \qquad l \geq |k|$$

$$-j \leq m \leq j$$

$$\mathrm{Ind}_{U(1)}^{SU(2)}(\rho_k) \cong \bigoplus_{\substack{l \geq |k| \\ l-k \in \mathbb{Z}}} V_l$$

$$\mathrm{Heis}(V,\Omega) \supset L$$

But ! SvN rep. $\quad \downarrow$

$$L'$$

$k : G \times G \longrightarrow U(1)$ "symplectic form"

$$1 \to U(1) \to \widetilde{G} \longrightarrow G \longrightarrow 1$$

$$\underbrace{\mathrm{SympAut}(G)}_{\text{preserve } k} \subset \mathrm{Aut}(G) \ni \alpha$$

$\alpha^* k$ - pull back function

$$(\alpha^* k)(g_1, g_2) := k(\alpha(g_1), \alpha(g_2))$$

$$\mathrm{SympAut} = \{ \alpha \mid \alpha^* k = k \}.$$

e.g. (V,Ω) $\quad$ SympAut = Symplectic Group.

$$1 \to U(1) \to \widetilde{G} \to G \to 1$$

Corresponding auts of $\widetilde{G}$? $\dashrightarrow \mathrm{Aut}(G)$

In general (not nec. Hei's G not. nec. Abelian)

$\pi : \widetilde{G} \to G$ homomorphism

$\alpha \in \mathrm{Aut}(G)$

we say $\widetilde{\alpha}$ is a <u>lift</u> of α if

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\pi"] \arrow[d, dashed, "\widetilde{\alpha}"] & G \arrow[d, "\alpha"] \\ \widetilde{G} \arrow[r] & G \end{tikzcd}$$

$$\circledast \quad \pi(\tilde{\alpha}(\tilde{g})) = \alpha(\pi(\tilde{g})) \quad \forall \tilde{g} \in \tilde{G}$$

Back to extensions

$$1 \to A \to \tilde{G} \to G \to 1$$

G, A Abelian : Written additively

Given $\alpha \in \mathrm{Aut}(G)$ can we lift it to $\tilde{\alpha} \in \mathrm{Aut}(\tilde{G})$
$$\tilde{\alpha} = T_\alpha$$

Lifting means

$$\boxed{T_\alpha(a,g) = (\xi_\alpha(a,g), \alpha(g))}$$

→ general solution to $\circledast$

Now T_α has to preserve the group law on $\bar{G}$

$$(a_1, g_1) \cdot (a_2, g_2)$$

$$= \left(a_1 + a_2 + f(g_1, g_2),\ g_1 + g_2\right)$$

$\Longrightarrow$ Constraint on $\xi_\alpha(a, g)$

$$T_\alpha(a, g) = \left(a + \tau_\alpha(g),\ \alpha(g)\right)$$

where

$$\boxed{\alpha^* f - f = \tau_\alpha(g_1 + g_2) - \tau_\alpha(g_1) - \tau_\alpha(g_2)}$$

$\Longrightarrow$ $\alpha^* k = k$

$$\boxed{\mathrm{Aut}_0(G)} \subset \mathrm{SymplAut}(G)$$

$\alpha^*[f] = [f] \quad [f] \in H^2(G, A)$

Example: Heis $(\mathbb{R}\oplus\mathbb{R})$

i.e. $\hat{q}, \hat{p}$ of Q.M.

$$\mathrm{SympAut}(\mathbb{R}\oplus\mathbb{R}) = Sp(2,\mathbb{R})$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^{tr} J A = J$$

$$A^{tr} J A = (ad-bc)\, J$$

for <u>any</u> 2×2 matrix

$$Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$$

(For $n>1$ $SL(2n,\mathbb{R})$ and $Sp(2n,\mathbb{R})$ are completely different.)

$$f\left((\alpha_1\beta_1)(\alpha_2\beta_2)\right) = \frac{\hbar}{2}(\alpha_1\beta_2 - \alpha_2\beta_1)$$

$$\begin{array}{cc} \cap & \cap \\ \mathbb{R}\oplus\mathbb{R} & \mathbb{R}\oplus\mathbb{R} \end{array}$$

$$\boxed{g^* f = f} \qquad g \in Sp(2,\mathbb{R})$$

So $\tau_g(\alpha,\beta) = 0$.

So symplectic group lifts to a group of aut's of $Heis(\mathbb{R}\oplus\mathbb{R})$ in a completely straightforward way.

<u>However</u>: The action on the SvN rep is subtle.

Lie algebra

$sp(2,\mathbb{R}) = sl(2,\mathbb{R})$

$= \{2\times 2$ real traceless matrices$\}$

Spanned by

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

$[h,e] = -2e$ $[e,f] = h$ $[h,f] = 2f$

Represent the Lie algebra:

$\hat{e} = \rho(e)$ etc.

$\hat{e} = \frac{i}{2\hbar}\hat{p}^2$ $\hat{h} = \frac{i}{2\hbar}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)$

$\hat{f} = \frac{i}{2\hbar}\hat{q}^2$ Check commutators work.

Now consider $U(1) \subset SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$

$$\exp(\theta(e+f)) \quad \boxed{\theta \sim \theta + 2\pi}$$

$$= \cos\theta + \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

maximal compact abelian subgroup.

How is ~~it~~ represented on $L^2(\mathbb{R})$?

$\exp(\theta(\hat{e}+\hat{f}))$ operates on

$$\hat{e}+\hat{f} = \frac{i}{2}(\hat{p}^2 + \hat{q}^2)$$

Spectrum is $\{ i(n+\frac{1}{2}) \mid n \geq 0 \}$

Because of the $\frac{1}{2}$ in the gnd state energy

$\exp\left(\theta\left(\hat{e}+\hat{f}\right)\right)$ has periodicity

$\theta \sim \theta + 4\pi$

$\exp\left(x\hat{e} + y\hat{h} + z\hat{f} \right)$

x, y, z real generates a double cover of $Sp(2,\mathbb{R})$

"metaplectic group"

$$1 \to \mathbb{Z}_2 \to \underbrace{Mpl(2,\mathbb{R})}_{\text{has } \underline{\text{NO}} \text{ finite-dim'l faithful rep's!}} \longrightarrow Sp(2,\mathbb{R})$$

$Mpl(2,\mathbb{R})$ is an example of a Liegroup that is NOT a subgroup of $GL(N,\mathbb{R})$ for any N !

Irred. F.D. Reps of $sl(2,\mathbb{R})$

$SL(2,\mathbb{R})$

V can show $\exists$ v_0

$\rho(e)v_0 = 0 \qquad \rho(h)v_0 = -N v_0$

N some integer.

$v_0, \rho(f)v_0, \ldots, \rho(f)^N v_0$

$\rho(h)$ ev $\quad -N \quad -N+1 \quad \ldots \quad +N.$

$sl(2,\mathbb{R}) \neq su(2)$ as real Lie algebras

BUT

$$sl(2,\mathbb{R}) \otimes \mathbb{C} \cong su(2) \otimes \mathbb{C}$$
$$\cong sl(2,\mathbb{C})$$

$$e = iT^1 - T^2$$
$$h = -2iT^3$$
$$f = -iT^1 - T^2$$

$T^k = -\frac{i}{2}\sigma^k$ generate $su(2)$ as a real Lie algebra

We've constructed the irreps of $SU(2)$.

$$e+f \longleftrightarrow i J^1$$

In <u>any</u> f.d. rep. of $su(2)$

$\rho(e)+\rho(f)$ is diagonalizable

has eigenvalues of the form

$i\ell$, $\ell \in \mathbb{Z}$. $\Longrightarrow$ In

finite-dim'l rep's

$$\exp\left(\theta\left(\rho(e)+\rho(f)\right)\right)$$

has period $\theta \sim \theta + 2\pi$

$\Longrightarrow$ $Mpl(2,\mathbb{R})$ cannot be

faithfully represented in finite

dimensions.

$\exp \theta(\hat{e}+\hat{f})$

very interesting 1-parameter family of operators

period $\theta \sim \theta + 4\pi$

$\theta = \pi/2$ Fourier transform

$$\left(e^{\theta(\hat{e}+\hat{f})}\psi\right)(x) \overset{\theta=\pi/2}{=} \hat{\psi}(x)$$

$$= \frac{e^{i\pi/4}}{\sqrt{2\pi}} \int e^{ixy}\psi(y)\,dy$$

$$\left(e^{\pi(\hat{e}+\hat{f})}\psi\right)(x) = \psi(-x)$$

Note That Square of Fourier transform is NOT the identity.

$\psi(x) \longrightarrow \psi(-x)$

Above shows that F.T.
has order 4 not 2.
And many other things.....